

Hermeneutics of the differential calculus in eighteenth-century Europe: from the *Analyse des infiniment petits* by L'Hôpital (1696) to the *Traité élémentaire de calcul différentiel et de calcul intégral* by Lacroix (1802)¹²

Mónica Blanco Abellán
Departament de Matemàtica Aplicada III
Universitat Politècnica de Catalunya

In the history of mathematics it has not been unusual to assume that the communication of mathematical knowledge among countries flowed without constraints, partly because mathematics has often been considered as “universal knowledge”. Schubring,³ however, does not agree with this view and prefers referring to the basic units of communication, which enable the common understanding of knowledge. The basic unit should be constituted by a common language and a common culture, both interacting within a common national or state context. Insofar this interaction occurs within a national educational system, communication is here potentially possible. Consequently Schubring proposes comparative analysis of textbooks as a means to examine the differences between countries with regard to style, meaning and epistemology, since they emerge from a specific educational context.

Taking Schubring's views as starting point, the aim of this paper is to analyze and compare the mathematical development of the differential calculus in France, Germany, Italy and Britain through a number of specific works on the subject, and within their corresponding educational systems. The paper opens with an outline of the institutional framework of mathematical education in these countries. In order to assess the mathematical development of the works to be analyzed, the paper proceeds with a sketch of the epistemological aspects of the differential calculus in the eighteenth century. Settling the works in their corresponding contexts is the next step. As the title

¹ This paper summarizes my PhD thesis presented in 2004 within the *Inter-university PhD Program in History of Science* (UAB-UB), coordinated by the Centre d'Estudis d'Història de la Ciència (CEHIC), Barcelona.

² I want to express my gratitude to my supervisor, Dr. Josep Pla (Universitat de Barcelona), and to the Centre d'Estudis d'Història de les Ciències (Universitat Autònoma de Barcelona). To Dr. Gert Schubring (Universität Bielefeld) I am especially indebted for his encouragement throughout the period of research, for his advice and for stimulating discussions on the subject of my study. I wish to thank the Institut für Didaktik der Mathematik (Universität Bielefeld), where part of the research for the present work was done.

suggests, the comparative analysis spans from the publication of the first systematical treatise on differential calculus, the *Analyse des infiniment petits* by L'Hôpital,⁴ to that of the *Traité élémentaire de calcul différentiel et de calcul intégral* by Lacroix.⁵ Accordingly some examples are introduced to illustrate the analysis, which will lead to the final conclusions.

Institutional Framework

We may start reviewing broadly the institutional framework regarding mathematical education in the eighteenth century. The countries brought into discussion are France, Germany, Italy and Britain.

France and Germany

In the aforementioned study Schubring chooses France as reference point owing to its dominant role in mathematics in this period. Since Germany succeeded France as a centre of mathematics in the 1830s, he focuses first on communications between the French and the German mathematical communities. For this reason, my preference here is to discuss both countries together.

Several educational systems coexisted in eighteenth-century France. Before the Revolution, university education was mainly restricted to the *collèges*, run by religious orders, where mathematics was taught at a rather elementary level. By the 1750s, however, a well-developed network of *écoles militaires* had been established. Mathematics became a leading discipline within this context, where abstract theorizations were held in low esteem, the emphasis definitely being on applications. After the Revolution, another educational system emerged. Among the revolutionary reforms it stands out the establishment of the École Polytechnique in 1794, where mathematics was given a prominent role.

In Germany there was no national or cultural unity at all. After the Thirty Years' War, it had been split into hundreds of states of either Catholic or Protestant faith, each

³ Schubring, G. (1996). Changing cultural and epistemological views on mathematics and different institutional contexts in nineteenth-century Europe. In: Goldstein, et al. (eds.), *Mathematical Europe. Myth, History, Identity*, Paris, Éditions de la Maison des sciences de l'homme.

⁴ L'Hôpital, G. F. A. de (1696). *Analyse des infiniment petits pour l'intelligence des lignes courbes*, Paris.

⁵ Lacroix, S. F. (1802). *Traité élémentaire de Calcul différentiel et de Calcul intégral*, Paris.

of which had its own educational system. In northern German states of Protestant faith, mathematical studies were established within the philosophical faculties, independent of the secondary schools. Unlike the French institutional structures, the universities of the Protestant German states insisted on reflections on the foundations of science. Universities, especially Halle and Göttingen, represented a step towards professionalization. Professors held permanent positions and, besides their salaries, they received fees from their students. In fact mathematics professors were required to perform research tasks and to offer good teaching to attract larger numbers of students. By contrast, education in southern German states was under the strict domination of the Catholic order of the Jesuits, who resisted any social and educational reform. Hence, mathematics remained a marginal, somewhat elementary, subject. In contrast to France, military schools were not of great significance in Germany.

Aside from correspondence and academy exchanges, up to the 1790s mathematical exchanges between France and the German states were rare. This lack of mutual communication may be attributed to differences in their educational systems. The establishment of the École Polytechnique and the congress on the reformed system of weights and measures (1798-1799), among other factors, favoured an increasing interest in Germany on the progress in French mathematics. Yet, the interest was not exactly mutual. Actually there was hardly any translation of German works into French.⁶

As far as textbook production is concerned, France was not the leader, not even in calculus. Since 1794 a project of elementarizing mathematical knowledge was undertaken. Centrally prescribed books were thought to ensure homogeneity and high quality of teaching. To the purpose several *concours* were held to select the best basic textbooks for the general education system, the so-called *livres élémentaires*. The German states, specifically Prussia, illustrate the opposite situation: a non-centralist educational policy resulted in a huge textbook production.⁷

⁶ Schubring, *Changing cultural*, note 3, pp. 366-367.

⁷ Schubring, G. (1997). *Analysis of Historical Textbooks in Mathematics (Lecture Notes)*, Rio de Janeiro, PUC Rio de Janeiro, p. 84, table 2.

Italy

Despite being split into several states and city-states, religious homogeneity prevailed in eighteenth-century Italy, where the Catholic faith was accepted in every state. Catholic institutions, specifically the Jesuit ones, were in charge of the educational system, within which mathematics was taught at an elementary and traditional level, displaying no specialization. Academies and societies were more active in mathematical research, in general, and in the study of Leibnizian calculus, in particular, than the universities. Even at military schools, like the Royal School of Artillery at Turin, mathematical education was better than at universities. However, the acknowledgement that mathematics would be useful for the engineering works undertaken to prevent the floodings of the river Po involved a change towards specialization. This process culminated in 1771, when the first chair of Mathematics was established at the University of Ferrara. Such an institutionalization pointed to the future integration of the engineering instruction within the university context. In turn, this entailed a growing institutionalization of mathematics education, oriented towards applications.⁸

Britain

The essential peculiarity of British mathematics is by all means its traditional adherence to Newtonian calculus. However, despite the frequent remarks upon the isolationism of British mathematics, Schubring hints at the lack of detailed studies on the actual exchanges between British and continental European mathematics and at the need to analyze in detail some aspects of mathematical education within the British system.⁹

Turning to fluxional calculus, it was hardly known for the first three decades of the eighteenth century, most certainly because of the late publication of Newton's manuscripts and the misunderstandings that this circumstance entailed. Around the middle of the century, however, the group of *philomaths*¹⁰ and students at military academies (like the Royal Military Academy at Woolwich) and universities (like Edinburgh and Cambridge) set into practice Newton's calculus. In an attempt to give

⁸ See Schubring, G. (2002). Mathematics between propaedeutics and professional use: a comparison of institutional developments, *Enciclopedia Italiana*, Rome, Istituto dell'Enciclopedia Italiana, VI.

⁹ Schubring, *Changing cultural*, note 3, pp. 376-377.

¹⁰ On *philomaths* and fluxional calculus see Guicciardini, N. (1989). *The Development of Newtonian Calculus in Britain 1700-1800*, Cambridge, Cambridge University Press, p. 65.

shape to the fluxional calculus, the production of textbooks on fluxions dramatically soared in the period 1736-1758.¹¹

Epistemological Framework¹²

How did the authors analyzed lay the foundations of differential calculus in their respective works? To answer this question it is first necessary to refer to the epistemological views of Gottfried Wilhem Leibniz (1646-1716), Isaac Newton (1643-1727) and Leonhard Euler (1707-1783), since these impregnated all the works considered in this study.

Most of Leibniz's research was known through his correspondence, along with his close connection with academies and societies. Certainly, the first printed exposition of the differential calculus was a short paper by Leibniz in the *Acta Eruditorum* of 1684.¹³ In this paper Leibniz defined differences, stated – without proof – the basic rules of differentiation, and applied them to problems on tangents and singular points. Dealing with the characteristic triangle Leibniz became aware that finding the quadrature under a curve and determining the tangent to the curve were reciprocal operations. An essential point in his development was that a curve could be considered to be identical with an infinitangular polygon, that is, a polygon of infinitely many infinitely small sides. This naturally implied that the tangent could be taken for the extension of a side of the infinitangular polygon. Then, from similar triangles the formula for the subtangent, s , could be computed as follows: $\frac{dy}{dx} = \frac{y}{s}$.

How should the sides of the infinitangular polygon be chosen? Leibniz was reluctant to consider the sides to be equal, in considering the method would lack generality. Hence, he suggested to choose the progression of variables¹⁴ (i.e. abscissas) according to the nature of the problem to be solved. In this way, the choice of a convenient progression would free the problem from annoying calculations.

¹¹Ibid.

¹² Here I outline epistemological aspects. Aside from them I have also discussed methodological and formal aspects, such as pedagogical skills, notation and structure in Blanco, M. (2004). *Hermenèutica del càlcul diferencial a l'Europa del segle XVIII*:... [<http://www.tdx.cbuc.es/>], ch. 8, and Blanco, M. (2006). Textbooks on Differential Calculus in 18th Century Europe: A Comparative Stylistic Analysis, *Journal of Data Science* (forthcoming).

¹³ Leibniz, G. W. (1684). Nova methodus pro maximis & minimis..., *Acta Eruditorum*.

Unlike Leibniz, however, in his *Institutiones calculi differentialis*¹⁵ of 1755 Euler considered the sequences of values as not induced by the infinitesimal polygon, but by a function of an independent variable. This variable played a crucial role with regard to higher-order differentiation. The differences between a sequence of values corresponding to the (independent) variable should be constant. If not, higher-order differentials would remain vague and meaningless. Thus Euler based the differential calculus upon the differential coefficient. As a new function, whose differential coefficient could be calculated in turn, the differential coefficient was to supply a sound foundation for higher-order differentiation.

On the other hand, between 1669 and 1676 Isaac Newton worked on his calculus, but from different foundational approaches. Newton's works did not start getting published until 1704, to claim priority in the Newton-Leibniz controversy on the invention of the new calculus. In his first tract, *De analysis per aequationes numero terminorum infinitas* (1669, published in 1711), Newton dealt with moments, which were infinitely small quantities, not differing essentially from Leibnizian differences. His *Methodus fluxionum et serierum infinitarum* (1671, published in 1736) introduced his characteristic notation and concepts on the theory of fluxions. Finally, Newton composed *De quadratura curvarum* (1676, published in 1704) in terms of the limit of the ratio of fluxions, the so-called prime and ultimate ratio. His reluctance to publish proved to be a source of misunderstanding for Newtonian calculus. Although Newton's research was already known by his correspondents, the main concepts of his fluxional method appeared for the first time in his *Principia Mathematica* (1687), where the algorithm of fluxions was not properly displayed.¹⁶

The epistemological guidelines for my comparative analysis of textbooks were suggested by my previous comparative analysis of L'Hôpital and Bernoulli.¹⁷ Did these textbooks present a tendency towards reflections on the foundations of differential calculus? Did they define infinitesimal differences, extrapolate from finite to infinitely small differences, reckon derivatives of functions, or determine fluxions? Likewise it was highly revealing to assess whether they included rigorous demonstrations, all the

¹⁴ Bos, H. (1974). Differentials, Higher-Order Differentials and the Derivative in the Leibnizian Calculus, *Archive for the History of Exact Sciences*, (14), pp. 41-42.

¹⁵ Euler, L. (1755). *Institutiones calculi differentialis*, Berlin.

¹⁶ Guicciardini, *Newtonian Calculus*, note 10, p. 1.

¹⁷ Blanco, M. (2001). Análisis de la discusión L'Hôpital-Bernoulli, *Cronos*, vol. 4 (1-2).

more so because the way the authors dealt with the status of higher-order differentiation followed naturally from the statement of fundamental concepts.

The epistemological approach of these works can be inferred from the theoretical issues they covered. The authors tending to a geometric approach based their proofs on Greek geometry. Among them it was not rare to meet the concept of motion as generator of curves. At the opposite end, a tendency towards algebraic language could be detected, which entailed the use of functions, the application of Taylor series expansion, and the characterization of the limit. Consequently, these views would affect the choice of coordinates depending on the specific nature of the curve.¹⁸

Textbooks on Differential Calculus in Eighteenth-Century Europe

This paper aims at analyzing the mathematical development and teaching of the differential calculus in eighteenth-century Europe by examining a wide array of works on the subject intended for the different educational systems discussed in the first section of this paper. Most of them are not acknowledged as “great books” of science. My preference is to focus on the “forgotten books”, as Topham puts it.¹⁹ Here I understand “forgotten” in the sense that most of the works of my study were not “canonical works”, and even some of them were not published until recently.

Works on differential calculus in the eighteenth century were intended for different kinds of audience. Some of them were originally addressed to beginners and to erudite learners in general (A). A second group was used in the university context (B). Finally, another group was related to military schools (C).²⁰ Obviously these groups were by no means disjoint sets with no elements in common and the real audience could differ from the original intended one. This paper, though, focuses on intended audiences, as described in the books’ prefaces.

¹⁸ Engelsmann, S. B. (1984). *Families of Curves and the Origins of Partial Differentiation* fully surveyed the studies of curves linked to the development of differential calculus. However, it does not discuss the curves in context and rather focused on the “great men”.

¹⁹ Topham, J. R. (2000). Scientific Publishing and the Reading of Science in Nineteenth-Century Britain: A Historiographical Survey and Guide to Sources, *Studies in History and Philosophy of Science*, 31 (4), pp. 566-567.

²⁰ Too technical a subject, calculus would not have attracted the interest of an audience beyond these three contexts. Here I am paraphrasing Guicciardini regarding the public of the calculus on fluxions in London and in the provinces at the beginning of the eighteenth century. See Guicciardini, *Newtonian Calculus*, note 10, p. 65.

The following subsections describe briefly the works to be analyzed, grouped by country. Though the present political and geographical map of Europe differs greatly from that of the eighteenth century, out of simplification the works chosen for the purpose of this study were grouped according to their current national identity, namely, France, Germany, Italy and Britain. A letter in brackets beside every work indicates the group in which they were more likely to be included (A, B, C).

France

Chronologically this paper begins with the Marquis de L'Hôpital (1661-1704), who in 1696 published the first textbook on differential calculus, the *Analyse des infiniment petits* [A]. This work relied largely upon the lectures that Johann Bernoulli (1667-1748) gave L'Hôpital between 1691 and 1692, which were not published until 1922.²¹ The *Analyse des infiniment petits* was widely read during the eighteenth century.²² L'Hôpital was introduced to Johann Bernoulli by Nicolas Malebranche (1638-1715). A member of the congregation of the Oratoire since 1660, Malebranche exerted a large influence on the development and spread of mathematics through the group he built up in Paris. Malebranche's role in the publication of L'Hôpital's textbook certainly contributed to the spread of Leibnizian calculus.

It was also Malebranche who encouraged Charles René Reyneau (1656-1728), another member of the Oratoire, to write and publish his *Analyse démontrée* (1708) [A],²³ which is one of the French works discussed in this paper. Both L'Hôpital's and Reyneau's works were intended for beginners.

Within the military educational system the figure of Étienne Bézout (1730-1783) stands out as a popular textbook writer on mathematics,²⁴ his audience being mainly the students of the various military institutions where he taught. Among his works, I will

²¹ Bernoulli, J. (1922). *Lectiones de calculo differentialium (1691-92)*, ed. P. Schafheitlin, Basel. There is an analysis and comparison of L'Hôpital's *Analyse* with Johann Bernoulli's *Lectiones* in Blanco, *Análisis de la discusión*, note 17.

²² At least, it was several times reissued in Paris in the 18th century, and also edited in Avignon (1768). Stone translated it into English (1730), but in terms of fluxions out of respect for Newton. Some Latin translations were issued in Vienne.

²³ See Gillispie, C. C., ed.. (1970-). *Dictionary of Scientific Biography*, New York, Charles Scribner's Sons.

²⁴ It was translated into English several times. These translations were used at schools and universities in the United States in the 19th century.

examine his *Cours de mathématiques à l'usage du corps de l'artillerie* (1799-1800) [C].²⁵

Late in the century the École Polytechnique was established as part of the revolutionary educational reforms. Both Joseph-Louis Lagrange (1736-1813) and Sylvestre François Lacroix (1765-1843) taught mathematics there and were as well involved in the commissions for choosing the best textbooks per discipline in the new system. Their elementary treatises on calculus, *Leçons sur le calcul des fonctions* (1800) [B] by Lagrange and *Traité élémentaire de calcul différentiel et de calcul intégral* (1802) [B] by Lacroix, are really a representative sample of *livres élémentaires*, since they were widely used at the École Polytechnique and other French institutions for decades.²⁶ It is not unreasonable to think that their involvement in the aforementioned commissions should have favoured the use of their treatises at that time. Both treatises were not translated into German until the nineteenth century. This circumstance confirms Schubring's statement on the scarce mutual exchange between France and Germany.²⁷

Germany

In the German states of Protestant faith research in mathematics was fostered at the universities. This is why this paper mainly focuses on authors related to either the University of Halle or the University of Göttingen. The German philosopher Christian Wolff (1679-1754), whose philosophical system was an adaptation of that of Leibniz, began his career as *Privatdozent* at the University of Leipzig. In 1723 he became professor of mathematics and natural philosophy at the University of Halle. During the first half of the eighteenth century his textbooks were most influential in northern German universities. His outstanding series *Anfangsgründe aller mathematische Wissenschaften* (1710) were translated into Latin in 1713, as *Elementa Matheseos Universae*, becoming an influential work in Europe. In this paper I will examine the volume *Elementa Analyseos* containing the chapter *Elementa Analyseos Infinitorum*

²⁵ See Schubring, *Changing cultural* note 3 and, *Historical Textbooks*, note 7.

²⁶ See Grattan-Guinness, I. (1990). *Convolution in French Mathematics*, Basel, Birkhäuser; Schubring, G. (1987). On the Methodology of Analysing Historical Textbooks: Lacroix as Textbook Author, *For the Learning of Mathematics*, 7 (3), and *Changing cultural*, note 3.

²⁷ Schubring, *Changing cultural*, note 3, p. 367.

[B].²⁸ Abraham G. Kästner (1719-1800) succeeded Johann A. Segner (1704-1777) as professor of mathematics at the university of Göttingen, when Wolff's post at the university of Halle went to Segner. I will study the second (revised) edition of Kästner's *Anfangsgründe der Analysis des Unendlichen* (1760) [B]. Segner's successor in Halle in 1778 was Wenceslaus J. G. Karsten (1732-1787), whose *Anfangsgründe der mathematischen Analysis und höhern Geometrie* (1786) [B] is analyzed in this paper.²⁹

The aforementioned German authors developed their practice of mathematics within the university context. This paper also deals with a volume addressed to the cadets of the Royal Prussian Artillery, *Anfangsgründe der Analysis des Unendlichen* (1770) [C], whose author, Georg F. Tempelhoff (1737-1807), was a member of this military academy.³⁰

Aside from Wolff, apparently the other three German authors would not have been translated, thus corroborating again Schubring with regard to the exchanges between France and Germany.

Italy

My survey of Italian textbooks on differential calculus begins with the *Instituzioni Analitiche* (1748) [A] by Maria Gaetana Agnesi (1718-1799). As she stated in her preface, this textbook was written in the Tuscan dialect for the sake of the Italian youth.³¹ In the eighteenth century women rarely committed themselves to scientific matters. This trend may account for Agnesi's popularity in the mathematical scenery of the century.³² Her text was influenced by her study of the works of L'Hôpital and Reyneau, along with her discussions with Jacopo Riccati (1676-1754) on topics related to calculus.

This group includes another work by Lagrange, since he was born in Turin, where he was appointed professor of mathematics at the Royal Artillery School in 1755. In fact

²⁸ I decided to study the Latin edition, since it seemed to be the one circulating among the European intellectual circles. Afterwards, however, I became aware of the fact that the Latin edition might not be a literal translation of the former German one. Hence, it would be fruitful to compare both in depth and to explore their differences.

²⁹ See Schubring, *Historical Textbooks*, note 7.

³⁰ There is hardly any reference about the life and work of Tempelhoff. I could only obtain some data by looking up into the *Deutsches Biographisches Archiv* edited by B. Fabian and W. Gorzny (1982-).

³¹ It was later translated into French (1775) and into English (1801).

³² See Truesdell, C. (1989). Maria Gaetana Agnesi, *Archive for History of the Exact Sciences*, 40.

his work, *Principj di analisi sublime* (1759) [C], was intended for the students of this School. Lagrange's later position at the École Polytechnique provides an example of the French influence on Italian mathematicians.³³

The Italian group concludes with the *Compendio d'analisi* (1775) [B] by Girolamo Saladini (1731-1813). This work is a revised translation of the *Institutiones Analyticae* (1765-67), written in Latin by the Jesuit Vincenzo Riccati (1707-1775) and Saladini himself.³⁴ Even though Truesdell regarded the *Institutiones Analyticae* as “the first Italian book on the calculus to come from university circles”,³⁵ to my knowledge neither of them was reissued nor translated.

Britain

The review on the institutional and epistemological framework of British mathematics made clear that, at the beginning of the eighteenth century, the texts on fluxions were rather awkward in their transmission of Newton's theory. In this paper I will examine one of the earliest treatises on fluxions, *An Institution of Fluxions* (1706) [A] by Humphry Ditton (1675-1715). Ditton's treatise does not contain a general study of the theory of curves; it only gives the basic rules and some applications.

As a consequence of the publication in 1734 of Berkeley's *The Analyst*, Colin Maclaurin (1698-1746) published his two-volume *A Treatise of Fluxions* (1742) [A]. Maclaurin aimed at defending Newton's calculus against Berkeley's attack, and his treatise was an attempt to provide Newton's calculus with systematic rigorous foundations. In his first book Maclaurin relied largely on Greek geometry and kinematic approach to fulfil his aim. But it is worth mentioning that the second part of the second book is devoted to physico-mathematical applications.

In the subsequent period to the debate on the foundations of Newtonian calculus (1736-1758), the fluxional treatise production soared dramatically. The works published then were especially concerned with applications of the fluxional methods to geometry and mechanics. To illustrate this period I studied a treatise by Thomas Simpson (1710-

³³ See Borgato, M. T.; Pepe, L. (1987). Lagrange a Torino (1750-1759) e le sue lezioni inedite nelle Reale Scuole di Artiglieria, *Bollettino di Storia delle Scienze Matematiche*, II.

³⁴ See Bagni, G. T. (1997). La didattica dell'Analisi matematica nel Settecento: le *Institutiones Analyticae* di V. Riccati e G. Saladini, *Periodico di Matematiche*, VII (4).

³⁵ The quotation is from Truesdell, *Agnesi*, note 32, n. 51.

1761), *The Doctrine and Application of Fluxions* (1750) [A, C]. As he claimed in the preface, Simpson aimed at helping beginners to understand fluxional methods. In 1743 Simpson obtained a chair of mathematics at the Royal Military Academy at Woolwich.³⁶ Thus his treatise also targetted students in this school. Simpson's works were very popular not only in Britain, but also in the Continent and in the United States.

While Ditton dealt with moments and prime and ultimate ratios, Maclaurin and Simpson alluded to fluxion as instantaneous velocity, that is, the velocity measured at any instant by means of the space that it would describe for a certain time, should the motion continue uniformly from that instant on.

One last remark must be added before proceeding further: all the works of this study were originally written in vernacular language. There are, however, two exceptions. On the one hand, Christian Wolff first published his *Anfangsgründe* in German, as part of Wolff's project to render all knowledge accessible to everyone. Yet, the Prussian authorities compelled Wolff to translate his work into Latin.³⁷ On the other hand, the work of Riccati and Saladini had been originally written in Latin. In eighteenth-century Italy the Jesuits played a dominant role in education. Actually Riccati was a Jesuit himself. This could explain the authors' preference for Latin in this work. Afterwards Saladini translated it into Italian, presumably for the benefit of the lay men and in connection with the growth of the Italian national spirit at the end of the century.

Some Examples

To illustrate the methodology that I applied in my study, I will discuss the following aspects: 1) the differentiation of the product, and 2) the determination of the tangent to the cycloid. As a basic rule of the differential calculus, the first aspect turned out to reveal the epistemological background of the authors involved. The second aspect is worth discussing to show how a problem which originated in the seventeenth century

³⁶ Before his position at Woolwich, he had been an itinerant teacher at the coffee-houses in London, and in 1754 he became the editor of the periodical *Ladies' Diary*. See Gillispie, Dictionary, note 23, and Guicciardini, Newtonian Calculus, note 10.

³⁷ See http://www.rasscass.com/templ/te_bio.php?PID=1229&RID=1.

was worked out, a century later, by means of the differential calculus, thus validating its power.

The Differentiation of the Product

The influence exerted by Leibniz, Newton and Euler can be traced in three different approaches to the differentiation of the product. Therefore the proofs provided by these three “great men”³⁸ open the exposition of each approach, respectively.

In his paper of 1684 Leibniz stated without proof the basic rules of differentiation. In particular the formula used by Leibniz for the differentiation of the product xv was:

$$d(xv) = xdv + vdx \quad (1)$$

In the introduction to his *Lectiones de calculo differentialium* Johann Bernoulli stated the following postulate, which introduced L’Hôpital’s *Analyse* as well:

A quantity diminished or increased by an infinitely small quantity is neither diminished nor increased.³⁹

Consequently, the following quantities in Figure 1 can be considered to be equal:

$$AP = Ap, PM = pm, Apm = APM,$$

and the section AMm equals the triangle $\triangle AMS$.

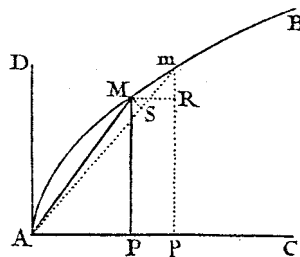


Figure 1. Introductory postulates in L’Hôpital (1696, Fig. 1)

³⁸ See note 19.

³⁹ Bernoulli, *Lectiones*, note 21, p. 3.

L'Hôpital relied on this postulate to formulate the rule for the differentiation of the product of xy . While quantity x increases, becoming $x + dx$, quantity y becomes $y + dy$. Considering dx as constant, the term $dxdy$ can be neglected since it is an infinitely small quantity with regard to ydx and xdy . A century later Bézout used the same approach in his *Cours de Mathématiques à l'usage du corps de l'artillerie*.⁴⁰

Wolff proved the formula for the difference of the product by taking the product xy for a rectangle. Then the differential of the product was the difference between the two rectangles thus generated:

$$ydx + xdy + dxdy \quad (2)$$

The differential of the rectangle ydx is $dxdy$, considering dx as constant. And so is the differential of xdy , considering now dy as constant. Hence the rectangle $dxdy$ vanishes with respect to ydx and xdy . Agnesi and Saladini considered motion as the origin of curves, but their proofs resembled that of Wolff's. Both eliminated $dxdy$ on the grounds that the expression $dxdy$ corresponds to a rectangle formed by two infinitesimals. Therefore, this rectangle is infinitely smaller than the other terms in the formula of the differentiation of the product. The rectangle here is an old-fashioned resource which emerges from the fact that the rectangular area equals the product of its sides. Thus, the use of a rectangle in this case provided calculus with geometric grounds.

In his *Principia mathematica* Newton⁴¹ considered A , B to be increased and decreased by $\frac{1}{2}a$ and $\frac{1}{2}b$ respectively, where a and b are the corresponding moments. The product of $A - \frac{1}{2}a$ and $B - \frac{1}{2}b$ yields

$$AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab \quad (3)$$

Likewise, the product of $A + \frac{1}{2}a$ and $B + \frac{1}{2}b$ yields

$$AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab \quad (4)$$

When subtracting (3) from (4) one obtains the moment of the original rectangle AB , namely $aB + bA$. In doing so, Newton apparently avoided the use of second order fluxions. His approach was strongly criticized by Berkeley in 1734.

Reyneau proved the elimination of $dxdy$ by compensation of errors in his *Analyse démontrée*, exactly in the same way as Newton did. Reyneau followed very

⁴⁰ Bézout, É. (1799). *Cours de mathématiques à l'usage du Corps de l'Artillerie*, Paris, §9.

⁴¹ Newton, I. (1687). *Philosophiae Naturalis Principia Mathematica*, London, book II, section II, lemma II.

closely the dispute which arose in 1700 in the French *Académie des Sciences*, centered on the nature of infinitesimals that originated from the publication of L'Hôpital's textbook.⁴² Since his text was published after this episode, Reyneau might have attempted to avoid the use of second order differences. Wolff proceeded this way too in the German version of his work of 1750,⁴³ surprisingly enough since this later version was issued long after Berkeley's attack.

But then, did fluxionists prove the fluxion of the product by compensation of errors? Often Maclaurin's treatise is praised because of his attempt to render the fluxional method more rigorous. To validate the fluxional formulas he combined the method of exhaustion and the kinematic approach. In §707 Maclaurin proved the following proposition by means of the method of exhaustion:

The fluxion of the root A being supposed equal to a , the fluxion of the square AA will be equal to $2A \times a$.⁴⁴

From the next article one infers how Maclaurin gets the fluxion of the product AB :

The fluxions of A and B being supposed equal to a and b , respectively, the fluxion of $A + B$ will be $a + b$, the fluxion of $\overline{A + B}^2$, or of $AA + 2AB + BB$, will be $2 \times \overline{A + B} \times \overline{a + b}$ or $2Aa + 2Bb + 2Ba + 2Ab$, by the last article [§707]. The fluxion of $AA + BB$ is $2Aa + 2Bb$, by the same; consequently the fluxion of $2AB$ is $2Ba + 2Ab$, and the fluxion of AB is $Ba + Ab$.⁴⁵

While Maclaurin employed the method of exhaustion, Ditton and Simpson turned again to the rectangle, combining it with the kinematic approach (see Figure 2).

⁴² On the dispute in the *Académie des Sciences* see Blay, M. (1986). Deux moments de la critiques du calcul infinitésimal: Michel Rolle et George Berkeley, *Revue d'histoire des sciences*, 39.

⁴³ Wolff, C. (1750). *Anfangs-Gründe aller mathematischen Wiessenschaften*, Halle, vol. IV, §16.

⁴⁴ Maclaurin, C. (1742). *A Treatise of Fluxions*, Edinburgh, T. W. and T. Ruddimans, p. 581.

⁴⁵ Ibid.

PROPOSITION II.

10. To find the Fluxion of the Product or Rectangle of two variable Quantities.

Conceive two Right-lines DE and FG, perpendicular to each other, to move, from two other Right - lines, BA and BC, continually parallel to themselves, and thereby generate the Rectangle DF. Let the Path of their Intersection, or the Loci of the Angle H, be the Line BHR; also let Dd (x) and Ff (y) be the Fluxions of the Sides BD (x) and BF (y), and let dm and fn, parallel to DH and FH, be drawn. Therefore, because the Fluxion of the Space or Area BDH is truly expressed by the Rectangle Dm ($= y \dot{x}$) and that of the Area, or Space BFH, by the Rectangle Fn, and equal Quantities have equal Fluxions, it follows that the Fluxion of the Rectangle $xy = DF$ ($= BDH + BFH$) is truly expressed by $y\dot{x} + x\dot{y}$. Q. E. I.

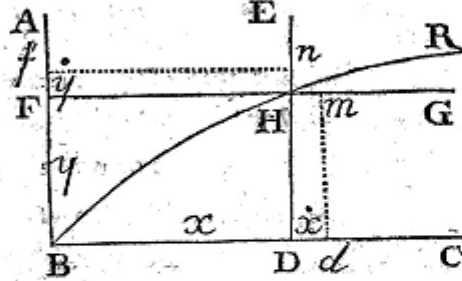


Figure 2. The fluxion of the product according to Simpson (1750, §10)

How did Euler (1755) compute the differential of the product $p \cdot q$? Given that p and q are functions of x , if p, q are increased by dp, dq , respectively, then:

$$(p + dp)(q + dq) = pq + pdq + qdp + dpdq \Rightarrow d.pq = pdq + qdp \quad (5)$$

The product $dpdq$ vanishes with respect to pdq and qdp since it is an infinitely small quantity of second order. Karsten and Lacroix proceeded as Euler did, except that they based their proof upon the limit of the ratio of differences.

In §41 of Lagrange's *Principj di analisi sublime*, dx and dy are first order infinitesimal differences. Then $dx dy$ is a second order infinitesimal, because it is the fourth proportional in $\frac{dx dy}{dy} = \frac{dx}{1}$. The finite difference of the product xy is $(x + dx)dy + ydx$. Then the term $dx dy$ vanishes when compared to $xdy + ydx$.

In order to get the differentiation of the product $Z = xy$, in his *Anfangsgründe*, §24, Kästner employed the equivalent expression

$$4Z = (x + y)^2 - (x - y)^2 \quad (6)$$

Then he applied the differentiation of the n -th power $Z = z^n$. Z becomes $Z + E$ when z is increased by e . Expanding the binomial Kästner comes to the conclusion that the ratio $E:e$ is the limit of $nz^{n-1}:1$, whenever e becomes infinitely small. Actually both ratios get as close as one may wish. Hence:

$$E = ne z^{n-1} \quad (7)$$

which yields

$$dZ = nz^{n-1}dz \quad (8)$$

Tempelhoff's proof relied on an unusual device. The differentiation of the product is obtained by applying logarithms. The differential of $u = axy + b$ has to be calculated. In §320 Tempelhoff proceeds as follows:

$$\begin{aligned} w &= u - b, \\ lw &= l(u - b) = la + lx + ly, \\ dlw &= dla + dlx + dly, \\ \frac{dw}{w} &= \frac{dx}{x} + \frac{dy}{y}, \\ dw &= \frac{w dx}{x} + \frac{w dy}{y}, \\ du &= \frac{(u - b) dx}{x} + \frac{(u - b) dy}{y}, \\ du &= ay dx + ax dy. \end{aligned} \quad (9)$$

In short, L'Hôpital, Wolff, Bézout, Agnesi and Saladini, like Leibniz, based the differential calculus upon infinitely small differences. To prove the differentiation of the product they stated that higher-order differences were negligible when compared with first order differences. When it comes to the fluxionists, instead of considering Newton's development, they introduced the kinematic approach in their proofs. To conclude, the use of functions and the limit of the ratio of differences shows how Euler influenced Lagrange (1759), Kästner and, to a greater extent, Karsten and Lacroix.

The Determination of the Tangent to the Cycloid

Let us outline first how the tangent to a curve was defined in the eighteenth century. While Leibniz identified an extended side of the infinitangular polygon with the tangent, Euler defined it as the limit of the secant lines, intersecting twice with the

curve. And for those who admitted the kinematic generation of curves, the tangent was the trajectory that a point would follow, if it was to be carried along uniformly thereafter.

Broadly speaking, L'Hôpital, Reyneau, Wolff, Agnesi, Saladini and Bézout tackled the problem following Leibniz's approach. Then, they manipulated with the differential triangle to get the tangent to a curve. Although Kästner's point of view was mostly influenced by Euler's, on the determination of tangents he still considered the curve as a polygon of infinitely many infinitesimal sides.

In his text of 1759 Lagrange's point of view differed from that of the Leibnizian group in that he introduced the concept of *limit*. He regarded the tangent as the limit of secant lines, as Euler himself did. Firstly he worked with the secant and finite differences, and then he considered the difference of abscissas approaching continually zero. The concept of limit is also found in the texts of Tempelhoff, Karsten and Lacroix. Tempelhoff computed the limit of the proportion of increments, whereas Karsten calculated the limit of the differential ratio, which actually coincides with the differential coefficient in Lacroix's approach. However, Lacroix obtained the tangent from similar triangles, considering two ordinates approaching each other "without end".

Lagrange's work of 1800 represented a step towards algebraization. It is well known that he tried to abandon the geometric views in his calculus.⁴⁶ Like Euler, he determined the tangent with the help of the series expansion of a function, developed till the first degree. As a matter of fact, the study of tangents is included in the same lesson where Lagrange studies the series expansion when cut from a certain term on.

How did fluxionists approach the problem of the determination of the tangent? In Ditton's treatise the tangent is the secant line when both its intersection points with the curve concur. From the evanescent triangle, which is generated by the uniform motion of the ordinate, Ditton arrived at the expression of the subtangent. As to Maclaurin, he proved the expression of the subtangent in the first book by means of the method of exhaustion and motion. Finally, Simpson took the tangent as the trajectory that a point would follow if it kept on moving with uniform motion. From the decomposition of the generator motion into two components and the evanescent triangle he reached the expression of the subtangent.

⁴⁶ See Grattann-Guinness, *Convolutions* note 26.

The treatment of algebraic and transcendental curves determined in turn the choice of coordinates. A geometric treatment of the curve led to look for coordinates according to the geometric nature of the curve. Thus the equation of transcendental curves became clearer and simpler. This was the usual procedure at the end of the seventeenth century. The tendency towards algebraization in the eighteenth century entailed an increasingly frequent use of orthogonal coordinates. In this case, the equation of algebraic curves benefited from such a choice.

Now I proceed to display how some of the authors of this study determined the tangent to the cycloid. I believe this curve is worth discussing, inasmuch as its study attracted mathematicians in the seventeenth century and on through the eighteenth.

So as to determine the subtangent of the cycloid in §15-17, L'Hôpital takes the abscissa over the generating circle, $x = AP$, with ordinate $y = PM$ (see Figure 3).

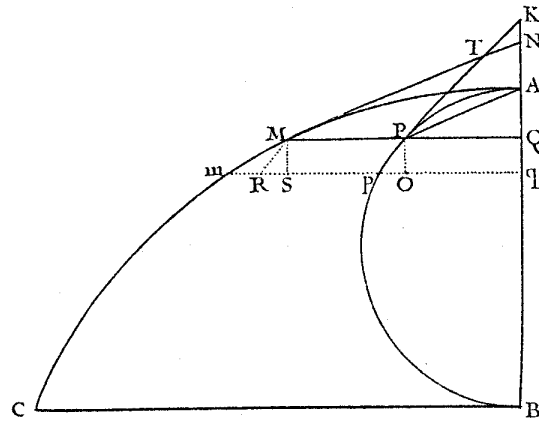


Figure 3. Determination of the tangent to the cycloid according to L'Hôpital (1696, Fig. 7)

By means of similar triangles L'Hôpital identified the elements of the expression for the subtangent

$$\frac{dy}{dx} = \frac{MP}{PT}, \quad (10)$$

$$PT = \frac{ydx}{dy}.$$

with the corresponding segments of the cycloid

$$x = \frac{ay}{b} \quad (11)$$

This approach produces a differential proportion,

$$dx = \frac{ady}{b}, \quad (12)$$

$$PT = \frac{ay}{b} = x.$$

and the analytical expression of the cycloid is avoided. Wolff, Agnesi and Simpson worked out the problem in a similar way.

Likewise Kästner takes into consideration the specific features of the curves for the transcendental case. Hence, from the definition of the cycloid itself, and taking the same coordinates, Kästner produces a differential equation expressing this curve, since it is not an algebraic curve.

On the contrary, in Karsten the choice of orthogonal coordinates prevails to such an extent that when he tackles a problem working with coordinates other than orthogonal, at the end he always introduces orthogonal coordinates by means of a convenient modification. The coordinates are the same used by Johann Bernoulli to solve this problem in his *Lectiones*.

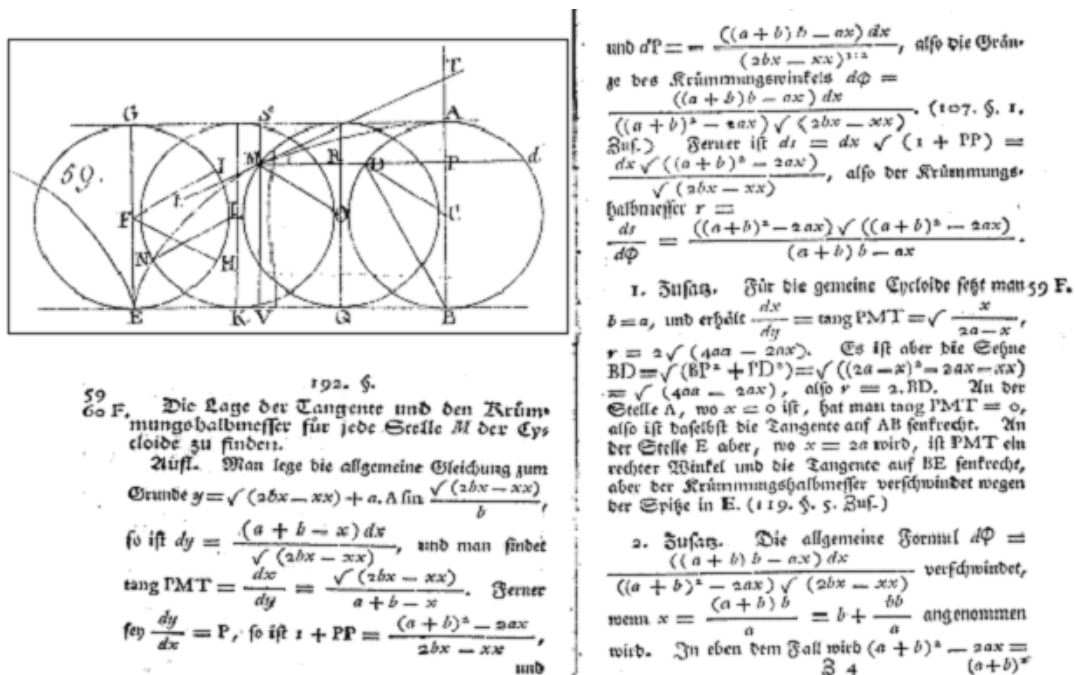


Figure 4. Determination of the tangent to the cycloid according to Karsten (1786, §192, Fig. 60)

Figure 4 shows that Karsten gets the equation:

$$y = \sqrt{(2ax - xx)} + a \cdot A \sin \frac{\sqrt{(2ax - xx)}}{a} \quad (13)$$

and calculates its differential ratio. Taking orthogonal coordinates as well, Riccati-Saladini and Lacroix considered the cycloid as a curve defined by differential equations.⁴⁷

Hence, in this section it has been noticed that L'Hôpital, Wolff, Agnesi and Simpson drew the tangent to the cycloid similarly. In their procedure similar triangles played a crucial role and the geometric features of the curve were taken into account on selecting the coordinates. By contrast, Karsten, Lacroix and Riccati-Saladini tended to take orthogonal coordinates, regardless the nature of the curve.

Conclusions

The above discussion on the differentiation of the product and the determination of the tangent to the cycloid helps to visualize groups of authors sharing epistemological features. Communication networks may account for these common features. In general, Leibnizian influence is easily traced in the works of L'Hôpital, Reyneau, Wolff, Agnesi and Saladini. How were Leibnizian principles conveyed? It has been mentioned that L'Hôpital and Reyneau belonged to the circle of Malebranche, who in turn was a follower of Leibniz, like Wolff. As far as Agnesi is concerned, she was connected with Leibnizian thinking through the study of the works of L'Hôpital and Reyneau. Besides she was lectured by Jacopo Riccati, whose son, Vincenzo, worked with Saladini.

Another communication network might justify the influence exerted by Euler on some authors. To begin with, Lagrange and Karsten were acquainted with Euler, the former through correspondence, the latter through academy meetings. Working within the same specified context might also result in epistemological similarities, as it happens with Kästner, Tempelhoff and Karsten. In this sense, Kästner's case is telling, since parts of his work share views with Wolff's, probably owing to the factor that both of them taught at the University of Halle. To conclude this network, Lacroix attempted to compose a coherent encyclopaedic work, which would provide a glimpse of the

⁴⁷ Riccati, V.; Saladini, G. (1765-67). *Institutiones Analyticae*, Bologna, Stamperia di Tommaso d'Aquino book II, ch. I, §10, and Lacroix, *Traité*, note 5, §102.

mathematical research of his time, and here Euler's contributions were explicitly included.

Underlying the epistemological features of these works some national trends can be made out, which agree with Schubring's views outlined at the beginning of this paper. German authors showed indeed a tendency towards reflections on foundations, while French authors were mainly concerned with applications, all the more so because their works were intended for beginners. My study of the Italian works also confirms their geometric approach of the subject, to such an extent that Agnesi, Riccati and Saladini admitted the kinematic generation of curves in their works. What is more, Riccati and Saladini illustrate the point that the differential calculus was not seriously tackled by the Italian universities till the 1770s.

With regard to British authors, it is pretty clear that they adhered to Newtonian theory. Yet, their exposition of fluxional calculus was by no means homogeneous, thus mirroring the confusion regarding the understanding of Newtonian calculus.

Even at military level this national patterns can be recognized. For instance, Tempelhoff confers great value to the exposition of foundations. By contrast, the level of Bézout's text was elementary because he did not regard as essential for an engineer's education matters such as a rigorous exposition of foundations. His point of view being similar to that of L'Hôpital, Wolff, Reyneau and Agnesi, his text is included in his volume on mechanics and hydrostatics, where the differential and integral calculus were considered to be of much help. On the fluxionist side, Simpson did not discuss largely on foundations either, but provided the reader with a wide range of applications of the fluxional calculus.

Both within the Italian and the French contexts, Lagrange turned out to be an exception, because of his innovative views on the calculus. In his work of 1759 he tended to abandon a pure geometrical exposition of the subject, thus anticipating his forthcoming algebraic approach.

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